

GAUSSIAN RATIONAL POINTS ON A SINGULAR CUBIC SURFACE

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ABSTRACT. Manin's conjecture predicts the asymptotic behavior of the number of rational points of bounded height on algebraic varieties. For toric varieties, it was proved by Batyrev and Tschinkel via height zeta functions and an application of the Poisson formula. An alternative approach to Manin's conjecture via universal torsors was used so far mainly over the field \mathbb{Q} of rational numbers. In this note, we give a proof of Manin's conjecture over the Gaussian rational numbers $\mathbb{Q}(i)$ and over other imaginary quadratic number fields with class number 1 for the singular toric cubic surface defined by $x_0^3 = x_1x_2x_3$.

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1. INTRODUCTION

Let $S \subset \mathbb{P}^3$ be the cubic surface defined over \mathbb{Q} by the equation

$$x_0^3 = x_1x_2x_3.$$

It is rational, toric and contains precisely three singularities and three lines. Over any number field K , its set of K -rational points is clearly infinite. Let H be the Weil height on S , defined as

$$H(\mathbf{x}) = \prod_{\nu \in M_K} \max_{j \in \{0, \dots, 3\}} \|x_j\|_{\nu}$$

where $\mathbf{x} = (x_0 : \dots : x_3) \in S(K)$ with $x_0, \dots, x_3 \in K$, the set of places of K is denoted as M_K and $\|\cdot\|_{\nu}$ is the (suitably normalized; see Section 3) norm at the place ν . The total number of K -rational points of bounded height on S is dominated by the number of easily countable points on the three lines. Therefore, we restrict our attention to K -rational points in the complement U of the lines on S .

Date: April 5, 2012.

2000 Mathematics Subject Classification. 11D45 (14G05, 14M25).

A much more general conjecture of Manin [FMT89] predicts in case of S that the number

$$N_{U,K,H}(B) = \#\{\mathbf{x} \in U(K) \mid H(\mathbf{x}) \leq B\}$$

of K -rational points of bounded height outside the lines behaves asymptotically as

$$N_{U,K,H}(B) \sim c_{S,K,H} B(\log B)^6,$$

as $B \rightarrow \infty$. A conjectural interpretation of the leading constant $c_{S,K,H} > 0$ was given by Peyre [Pey95] and refined by Batyrev and Tschinkel [BT98b].

Making use of the torus action on toric varieties to study the height zeta functions and to apply the Poisson formula, Manin's conjecture was proved for toric varieties over any number field by Batyrev and Tschinkel [BT98a]; see [BT98b, §5.3] for the application of this result to our cubic surface S .

For varieties without such an action of an algebraic group, an alternative approach using *universal torsors* was suggested by Salberger [Sal98]. He gave a second proof of Manin's conjecture over \mathbb{Q} in the case of split toric varieties; see [Sal98, Example 11.50] for its application to S .

For the singular cubic surface S as above, Manin's conjecture over \mathbb{Q} was also proved directly by Fouvry [Fou98], Heath-Brown and Moroz [HBM99], de la Bretèche [Bre98], de la Bretèche and Swinnerton-Dyer [BSD07] and Bhowmik, Essouabri, Lichtin [BEL07], using elementary or classical analytic number theoretic techniques and a parameterization of rational points closely related to universal torsors in some cases.

The basic example of a universal torsor applied to point counting is the following: To estimate the number

$$N_{\mathbb{P}^n, \mathbb{Q}, H}(B) = \#\{\mathbf{x} \in \mathbb{P}^n(\mathbb{Q}) \mid H(\mathbf{x}) \leq B\}$$

of rational points of bounded height in n -dimensional projective space \mathbb{P}^n , the natural first step is the observation that any such \mathbf{x} is represented uniquely up to sign by an $(n+1)$ -tuple of coprime integers (x_0, \dots, x_n) subject to the condition $\max\{|x_0|, \dots, |x_n|\} \leq B$. Geometrically, this corresponds to the fact that the open subset $\mathbb{A}^{n+1} \setminus \{0\}$ of $(n+1)$ -dimensional affine space is a universal torsor over \mathbb{P}^n .

Based on this, Schanuel [Sch64], [Sch79] proved Manin's conjecture for projective spaces over arbitrary number fields. Over number fields other than \mathbb{Q} , no other proof of Manin's conjecture via universal torsors is known to us.

The purpose of this note is to begin the generalization of universal torsor techniques from \mathbb{Q} to more general number fields. A first candidate is Manin's conjecture for the toric cubic surface S over the field $\mathbb{Q}(i)$ of Gaussian rational numbers because its class number is 1 and its ring of integers contains only finitely many units. It turns out that it is not too hard to generalize from $\mathbb{Q}(i)$ to the following setting:

Theorem. *Let K be an imaginary quadratic number field whose class number is 1, let w_K be the number of units in its ring of integers \mathcal{O}_K , and let d_K be the square root of the absolute value of its discriminant (cf. Table 1). Let $S \subset \mathbb{P}^3$ be the cubic surface defined by $x_0^3 = x_1 x_2 x_3$. Let U be the*

complement of the three lines on S . Then

$$N_{U,K,H}(B) \sim c_{S,K,H} B(\log B)^6 + O(B(\log B)^5)$$

as $B \rightarrow \infty$, with

$$c_{S,K,H} = \frac{2^7 \pi^9}{6! w_K^7 d_K^9} \prod_p \left(1 - \frac{1}{\|p\|_\infty}\right)^7 \left(1 + \frac{7}{\|p\|_\infty} + \frac{1}{\|p\|_\infty^2}\right)$$

where the product runs over all primes in \mathcal{O}_K up to units.

n	-1	-2	-3	-7	-11	-19	-43	-67	-163
w_K	4	2	6	2	2	2	2	2	2
d_K	2	$2\sqrt{2}$	$\sqrt{3}$	$\sqrt{7}$	$\sqrt{11}$	$\sqrt{19}$	$\sqrt{43}$	$\sqrt{67}$	$\sqrt{163}$

TABLE 1. $K = \mathbb{Q}(\sqrt{n})$ with class number $h_K = 1$.

We will see in Section 3 that this result agrees with the conjectures of Manin, Peyre, Batyrev and Tschinkel.

Recently, Frei [Fre12] generalized our work to arbitrary number fields, removing our restriction to class number 1 and finite groups of units in the ring of integers.

Acknowledgements. The authors are grateful to Tim Browning and the referee for helpful remarks. The first named author was supported by grant 200021_124737/1 of the Schweizer Nationalfonds and by grant DE 1646/2-1 of the Deutsche Forschungsgemeinschaft.

2. GEOMETRY

In this section, we collect some facts on the geometry of our singular cubic surface S . The construction of its minimal desingularization as a blow-up of the projective plane in six points will be used in Section 4 to construct a parameterization of the K -rational points by integral points on a universal torsor.

Let \mathcal{S} be the model of S over \mathbb{Z} defined by the equation $x_0^3 = x_1 x_2 x_3$ in $\mathbb{P}_{\mathbb{Z}}^3$. We will consider the minimal desingularization $\tilde{\mathcal{S}}$ of \mathcal{S} , which is obtained from $\mathbb{P}_{\mathbb{Z}}^2$ by a sequence of six blow-ups of points. All statements below will be true not only over \mathbb{Z} but, suitably rephrased, over any field. In what follows, all statements involving variables j, k, l are meant to hold for all

$$(j, k, l) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}.$$

The surface \mathcal{S} is singular, with precisely three singularities

$$p_1 = (0 : 1 : 0 : 0), \quad p_2 = (0 : 0 : 1 : 0), \quad p_3 = (0 : 0 : 0 : 1).$$

They are rational double points of type \mathbf{A}_2 in the **ADE**-classification. It contains precisely three lines $\ell_j = \{x_0 = x_j = 0\}$ through p_k and p_l .

The surface \mathcal{S} is toric. Indeed, an action of a two-dimensional torus on \mathcal{S} is given by

$$\mathbb{G}_{\mathfrak{m}, \mathbb{Z}}^2 \times \mathcal{S} \rightarrow \mathcal{S}, \quad (\mathbf{t}, \mathbf{x}) \mapsto \mathbf{t} \cdot \mathbf{x} = (x_0 : t_1 x_1 : t_2 x_2 : (t_1 t_2)^{-1} x_3),$$

giving an isomorphism from $\mathbb{G}_{m,\mathbb{Z}}^2$ to the open dense orbit

$$\mathcal{U} = \mathcal{S} \setminus (\ell_1 \cup \ell_2 \cup \ell_3) = \{\mathbf{x} \in \mathcal{S} \mid x_0 x_1 x_2 x_3 \neq 0\}$$

of $(1 : 1 : 1 : 1)$, say. The corresponding fan can be found in Figure 1.

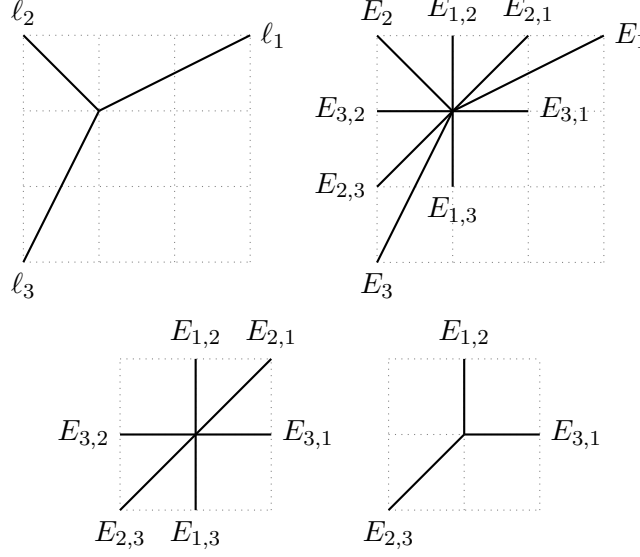


FIGURE 1. Fans of $\mathcal{S}, \tilde{\mathcal{S}}, \tilde{\mathcal{S}}_1, \mathbb{P}_{\mathbb{Z}}^2$, respectively.

Resolving the singularity p_j gives two exceptional divisors $E_{k,l}$ (meeting the strict transform E_l of ℓ_l) and $E_{l,k}$ (meeting the strict transform E_k of ℓ_k). We obtain the minimal desingularization $\pi : \tilde{\mathcal{S}} \rightarrow \mathcal{S}$, where the Picard group of $\tilde{\mathcal{S}}$ is free of rank 7. The six curves $E_{j,k}, E_{k,j}$ are (-2) -curves (rational curves with self-intersection number -2), while the three transforms E_j of the lines are (-1) -curves (rational curves with self-intersection number -1). There are no other negative curves (rational curves with negative self-intersection number) of $\tilde{\mathcal{S}}$. The negative curves correspond precisely to the rays in the fan of $\tilde{\mathcal{S}}$ in Figure 1.

The surface \mathcal{S} is rational, via the birational map

$$\phi : \mathcal{S} \dashrightarrow \mathbb{P}_{\mathbb{Z}}^2, \quad \mathbf{x} \mapsto (x_0^2 : x_0 x_1 : x_1 x_2) = (x_2 x_3 : x_0^2 : x_0 x_2) = (x_0 x_3 : x_1 x_3 : x_0^2),$$

(where the three expressions coincide where they are defined), with inverse

$$\psi : \mathbb{P}_{\mathbb{Z}}^2 \dashrightarrow \mathcal{S}, \quad \mathbf{z} \mapsto (z_0 z_1 z_2 : z_1^2 z_2 : z_2^2 z_0 : z_0^2 z_1).$$

Indeed, ϕ and ψ restrict to isomorphisms between the open subsets $\mathcal{U} \subset \mathcal{S}$ and $\mathcal{V} = \{\mathbf{z} \in \mathbb{P}_{\mathbb{Z}}^2 \mid z_0 z_1 z_2 \neq 0\} \subset \mathbb{P}_{\mathbb{Z}}^2$.

Then the following diagram commutes, where $\pi_0 : \tilde{\mathcal{S}} \rightarrow \mathbb{P}_{\mathbb{Z}}^2$ is the blow-up of $\mathbb{P}_{\mathbb{Z}}^2$ in six points in *almost general position* [DP80].

$$\begin{array}{ccc} \tilde{\mathcal{S}} & & \\ \pi \downarrow & \searrow \pi_0 & \\ \mathcal{S} & \xrightarrow[\phi]{} & \mathbb{P}_{\mathbb{Z}}^2 \end{array}$$

More precisely, π_0 maps

- $E_1, E_{2,1}$ to $(1 : 0 : 0)$,
- $E_2, E_{3,2}$ to $(0 : 1 : 0)$,
- $E_3, E_{1,3}$ to $(0 : 0 : 1)$,
- $E_{2,3}, E_{3,1}, E_{1,2}$ to $\{z_0 = 0\}, \{z_1 = 0\}, \{z_2 = 0\}$, respectively.

Conversely, using the same symbol for divisors on $\tilde{\mathcal{S}}$ and their projections and strict transforms on $\mathbb{P}_{\mathbb{Z}}^2$ and the intermediate $\tilde{\mathcal{S}}_1$ (for example, on $\mathbb{P}_{\mathbb{Z}}^2$, we have $E_{2,3} = \{z_0 = 0\}$, $E_{3,1} = \{z_1 = 0\}$ and $E_{1,2} = \{z_2 = 0\}$) we obtain

$$\pi_0 : \tilde{\mathcal{S}} \xrightarrow{\pi_2} \tilde{\mathcal{S}}_1 \xrightarrow{\pi_1} \mathbb{P}_{\mathbb{Z}}^2,$$

where $\tilde{\mathcal{S}}_1$ is a smooth sextic del Pezzo surface, by

- blowing up the three points $E_{j,k} \cap E_{k,l} \in \mathbb{P}_{\mathbb{Z}}^2$ with exceptional divisors $E_{l,k}$, respectively, to obtain $\pi_1 : \tilde{\mathcal{S}}_1 \rightarrow \mathbb{P}_{\mathbb{Z}}^2$;
- blowing up the three points $E_{k,j} \cap E_{l,j} \in \tilde{\mathcal{S}}_1$ with exceptional divisors E_j , respectively, to obtain $\pi_2 : \tilde{\mathcal{S}} \rightarrow \tilde{\mathcal{S}}_1$.

This gives $\tilde{\mathcal{S}}$ with three (-1) -curves E_j and six (-2) -curves $E_{j,k}, E_{k,j}$. Contracting the (-2) -curves via the anticanonical map gives $\pi : \tilde{\mathcal{S}} \rightarrow \mathcal{S} \subset \mathbb{P}_{\mathbb{Z}}^3$.

3. THE LEADING CONSTANT

For a smooth Fano variety defined over a number field K , Peyre [Pey95, Conjecture 2.3.1] gave a conjectural interpretation of the leading constant in Manin's conjecture. This was generalized to Fano varieties with at worst canonical singularities by Batyrev and Tschinkel [BT98b, §3.4 Step 4]. We will see that our theorem agrees with this prediction.

We start by collecting all number theoretic notation we need for this section. Let (r_K, s_K) be the number of real resp. pairs of complex embeddings of K , and let $q_K = r_K + s_K - 1$. Let \mathcal{O}_K be its ring of integers. Let $\mathcal{O}_K^{\neq 0} = \mathcal{O}_K \setminus \{0\}$. Let w_K be number of roots of unity in \mathcal{O}_K , and let R_K be the regulator of K . Let d_K denote the square root of the absolute value of the discriminant of K .

The set M_K of places of K consists of the archimedean places $M_{K,\infty}$ and the non-archimedean places $M_{K,f}$. For $\nu \in M_K$, let K_ν be the completion of K at ν and, for $\nu \in M_{K,f}$, let \mathbb{F}_ν be the residue field. For any $\nu \in M_{K,f}$, we define a norm by $\|x\|_\nu = |N_{K_\nu/\mathbb{Q}_p}(x)|_p$ for all $x \in K_\nu$, where p the characteristic of \mathbb{F}_ν and $|\cdot|_p$ is the usual norm on \mathbb{Q}_p . For any $\nu \in M_{K,\infty}$ corresponding to a real embedding $\sigma : K \rightarrow \mathbb{R}$, we define $\|x\|_\nu = |\sigma(x)|$ for all $x \in K_\nu$, where $|\cdot|$ is the usual absolute value on \mathbb{R} . For any $\nu \in M_{K,\infty}$ corresponding to a pair of complex embeddings σ, σ' , we define $\|x\|_\nu = |\sigma(x)|^2$, where $|\cdot|$ is the usual absolute value on \mathbb{C} .

We compute the expected asymptotic behavior of $N_{U,K,H}(B)$ with respect to the very ample anticanonical metrized sheaf $-\mathcal{K}_S = (-K_S, \|\cdot\|_\nu)$ [BT98b, Definition 3.1.3], where the family of ν -adic metrics corresponds to our anticanonical height function H .

We use the minimal desingularization $\pi : \tilde{S} \rightarrow S$ and its integral model $\pi : \tilde{\mathcal{S}} \rightarrow \mathcal{S}$ constructed in Section 2. As the \mathbf{A}_2 -singularities on S are rational double points, we have $\pi^*(K_S) = K_{\tilde{S}}$. Therefore, the $-\mathcal{K}_S$ -index [BT98b, Definition 2.2.4] of S is 1, and the $-\mathcal{K}_S$ -rank [BT98b, Definition 2.3.11] of S is $\text{rk Pic}(\tilde{S}) = 7$. Hence the expected asymptotic formula according to

Manin's conjecture is (in the notation of [BT98b, §3.4 Step 4])

$$N_{U,K,H}(B) = \frac{\gamma_{-\kappa_S}(U)}{6!} \delta_{-\kappa_S}(U) \tau_{-\kappa_S}(U) B(\log B)^6 (1 + o(1)).$$

The cohomological factor of the expected leading constant is

$$\delta_{-\kappa_S}(U) = \#H^1(\text{Gal}(\overline{\mathbb{Q}}/K), \text{Pic}(\tilde{S}_{\overline{\mathbb{Q}}})) = 1,$$

as $\text{Gal}(\overline{\mathbb{Q}}/K)$ acts trivially on $\text{Pic}(\tilde{S}_{\overline{\mathbb{Q}}})$ since \tilde{S} is split over K .

The factor $\gamma_{-\kappa_S}(U)/6!$ [BT98b, Definition 2.3.16] is simply $\alpha(\tilde{S})$ as in [Pey95, Définition 2.4]. By [DJT08, Theorem 1.3], we have

$$\frac{\gamma_{-\kappa_S}(U)}{6!} = \alpha(\tilde{S}) = \frac{\alpha(S_0)}{\#W(3\mathbf{A}_2)} = \frac{1}{120 \cdot (3!)^3} = \frac{1}{36 \cdot 6!},$$

where S_0 is a smooth cubic surface with $\alpha(S_0) = 1/120$ by [Der07, Theorem 4] and $W(3\mathbf{A}_2)$ is the Weyl group of the root system $3\mathbf{A}_2$ associated to the singularities of S .

Next, we compute the Tamagawa number

$$\tau_{-\kappa_S}(U) = \lim_{s \rightarrow 1} (s-1)^7 L(s, \text{Pic}(\tilde{S}_{\overline{\mathbb{Q}}})) \int_{\tilde{S}(\mathbb{A}_K)} \omega_{-\kappa_S}$$

[BT98b, Definition 3.3.10], where the set $\tilde{S}(\mathbb{A}_K)$ of adelic points coincides with the closure of $\tilde{S}(K)$ in it since \tilde{S} satisfies weak approximation. Here, we have used that every non-archimedean valuation ν is a good valuation in the sense of [BT98b, Definition 3.3.5] since the reduction of the model \tilde{S} of \tilde{S} at any finite place of K is a smooth projective variety.

Since \tilde{S} is split, the Frobenius morphism associated to every non-archimedean place ν corresponding to a prime ideal \mathfrak{p} acts trivially on $\text{Pic}(\tilde{S}_{\overline{\mathbb{F}}_\nu})$ of rank 7. Therefore, $L_\nu(s, \text{Pic}(\tilde{S}_{\overline{\mathbb{Q}}})) = (1 - \mathfrak{N}\mathfrak{p}^{-s})^{-7}$ (cf. [Pey95, §2.2.3]), and $L(s, \text{Pic}(\tilde{S}_{\overline{\mathbb{Q}}})) = \prod_{\nu \in M_{K,f}} L_\nu(s, \text{Pic}(\tilde{S}_{\overline{\mathbb{Q}}})) = \zeta_K(s)^7$. So

$$\lim_{s \rightarrow 1} (s-1)^7 L(s, \text{Pic}(\tilde{S}_{\overline{\mathbb{Q}}})) = \lim_{s \rightarrow 1} (s-1)^7 \zeta_K(s)^7 = \left(\frac{2^{r_K} (2\pi)^{s_K} h_K R_K}{w_K d_K} \right)^7$$

by the analytic class number formula.

Furthermore, by [BT98b, Definition 3.3.9],

$$\int_{\tilde{S}(\mathbb{A}_K)} \omega_{-\kappa_S} = d_K^{-\dim(\tilde{S})} \prod_{\nu \in M_K} \lambda_\nu^{-1} d_\nu(U)$$

where $\lambda_\nu = L_\nu(1, \text{Pic}(\tilde{S}_{\overline{\mathbb{Q}}}))$ for all (good) non-archimedean places and $\lambda_\nu = 1$ for the archimedean places. It remains to compute the local densities $d_\nu(U)$ defined in [BT98b, Remark 3.3.2].

We compute the archimedean densities on the open subset $U = \{x_0 \neq 0\}$ of S , defined by the cubic equation $f(x_0, \dots, x_3) = x_0^3 - x_1 x_2 x_3$, as

$$d_\nu(U) = \int_{S(K_\nu)} \omega_{-\kappa_S}(g) = \begin{cases} 36, & \nu \text{ real,} \\ 36\pi^2, & \nu \text{ complex.} \end{cases}$$

Indeed, we apply [Pey95, Lemme 5.4.4] and see via the birational morphism $\rho : U \rightarrow \mathbb{A}^2$ defined by $(x_0 : x_1 : x_2 : x_3) \mapsto (x_1/x_0, x_2/x_0)$ that

$$d_\nu(U) = \int_{(K_\nu^\times)^2} \frac{1}{\max\{1, \|y_1\|_\nu, \|y_2\|_\nu, \|(y_1 y_2)^{-1}\|_\nu\} \cdot \| - y_1 y_2 \|_\nu} dy_{1,\nu} dy_{2,\nu}.$$

A straightforward computation gives the values above. For complex ν , the Haar measure $dy_{i,\nu}$ on K_ν is normalized as *twice* the usual Lebesgue measure obtained from regarding $K_\nu \cong \mathbb{C}$ as \mathbb{R}^2 , as in [Pey95, §1.1]. For real ν , the value $d_\nu(S) = 36$ can also be found in [BT98b, §5.3].

As every non-archimedean place ν corresponding to a prime ideal \mathfrak{p} is good, we can apply [BT98b, Theorem 3.3.7] to compute

$$d_\nu(U) = \frac{\#\tilde{S}(\mathbb{F}_\mathfrak{p})}{\mathfrak{N}\mathfrak{p}^2} = 1 + \frac{7}{\mathfrak{N}\mathfrak{p}} + \frac{1}{\mathfrak{N}\mathfrak{p}^2},$$

since the norm $\mathfrak{N}\mathfrak{p}$ is the cardinality of the residue field \mathbb{F}_ν of K_ν . Indeed, for any finite field \mathbb{F}_q , the surface $\tilde{S}_{\mathbb{F}_q}$ is the blow-up of $\mathbb{P}_{\mathbb{F}_q}^2$ in six \mathbb{F}_q -rational points, and any such blow-up replaces one \mathbb{F}_q -rational point by a rational curve containing $q+1$ points over \mathbb{F}_q . Since $\#\mathbb{P}_{\mathbb{F}_q}^2(\mathbb{F}_q) = q^2 + q + 1$, we obtain the result. See also [Lou10, Lemma 2.3].

In total, the expected leading constant is

$$\frac{9^{q_K}}{4 \cdot 6!} \left(\frac{2^{r_K} (2\pi)^{s_K}}{d_K} \right)^9 \left(\frac{h_K R_K}{w_K} \right)^7 \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}} \right)^7 \left(1 + \frac{7}{\mathfrak{N}\mathfrak{p}} + \frac{1}{\mathfrak{N}\mathfrak{p}^2} \right).$$

For imaginary quadratic number fields K with class number $h_K = 1$, we have $(r_K, s_K) = (0, 1)$, so $q_K = 0$. Since the number w_K of units in \mathcal{O}_K is finite, its regulator R_K is 1. We denote the archimedean place as $\nu = \infty$. Let \mathbb{N}_K be a fundamental domain for $\mathcal{O}_K^{\neq 0}$ modulo the action of the units. We identify each prime ideal \mathfrak{p} with its unique generator $p \in \mathbb{N}_K$, with $\mathfrak{N}\mathfrak{p} = \|p\|_\infty$. We see that the expected leading constant of [BT98b] coincides with $c_{S,K,H}$ in our main theorem.

4. PASSAGE TO A UNIVERSAL TORSOR

We follow the strategy of [DT07]. This leads to a parameterization of rational points on S by integral points in \mathbb{A}^9 that is similar to the one used in [HBM99], but with a different set of coprimality conditions. We could construct coprimality conditions as in [HBM99], but we believe our conditions are more closely connected to the geometry of \tilde{S} and easier to work with. We note that our coprimality conditions are analogous to the ones obtained by Salberger [Sal98, 11.5] for toric varieties over \mathbb{Q} .

In the following, any statement involving j, k, l is meant to hold for all

$$(j, k, l) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}.$$

A parameterization of K -rational points on $U \subset S$ is obtained via the map ψ defined in Section 2. The isomorphism $\psi|_V : V \rightarrow U$ induces a map

$$\Psi_0 : (\mathcal{O}_K^{\neq 0})^3 \rightarrow S(K), \quad \mathbf{y} \mapsto \Psi_0(\mathbf{y}) = (\Psi_0(\mathbf{y})_0 : \dots : \Psi_0(\mathbf{y})_3)$$

where $\mathbf{y} = (y_{2,3}, y_{3,1}, y_{1,2})$ and

$$\Psi_0(\mathbf{y})_0 = y_{1,2} y_{3,1} y_{2,3}, \quad \Psi_0(\mathbf{y})_j = y_{j,k} y_{l,j}^2.$$

This induces a w_K -to-1 map from

$$\{(y_{2,3}, y_{3,1}, y_{1,2}) \in (\mathcal{O}_K^{\neq 0})^3 \mid H(\Psi_0(\mathbf{y})) \leq B, \gcd(y_{1,2}, y_{2,3}, y_{3,1}) = 1\}$$

to

$$N_0(B) = \{\mathbf{x} \in U(K) \mid H(\mathbf{x}) \leq B\}.$$

However, this parameterization is not good enough to start counting integral elements in a region in \mathcal{O}_K^3 because the height condition is not as easy as one might hope since $\gcd(y_{1,2}y_{2,3}y_{3,1}, y_{1,2}y_{3,1}^2, y_{2,3}y_{1,2}^2, y_{3,1}y_{2,3}^2)$ (taken here and always in \mathcal{O}_K) may be non-trivial even if $\gcd(y_{1,2}, y_{2,3}, y_{3,1}) = 1$.

Motivated by the construction of $\tilde{\mathcal{S}}$ as the blow-up of $\mathbb{P}_{\mathbb{Z}}^2$ in intersection points of certain divisors, we modify this as follows.

In the first step, let $y_{l,k} = \gcd(y_{j,k}, y_{k,l})$. Write $y_{j,k} = y'_{j,k}y_{k,j}y_{l,k}$. Then $\gcd(y'_{j,k}, y'_{k,l}) = \gcd(y_{l,k}, y_{k,j}) = \gcd(y_{k,j}, y'_{k,l}) = 1$. Now we drop the ' again for notational simplicity. We obtain a map

$$\Psi_1 : (\mathcal{O}_K^{\neq 0})^6 \rightarrow S(K), \quad \mathbf{y} \mapsto \Psi_1(\mathbf{y}) = (\Psi_1(\mathbf{y})_0 : \dots : \Psi_1(\mathbf{y})_3),$$

where $\mathbf{y} = (y_{1,2}, y_{2,1}, y_{1,3}, y_{3,1}, y_{2,3}, y_{3,2})$ and

$$\Psi_1(\mathbf{y})_0 = y_{1,2}y_{2,1}y_{1,3}y_{3,1}y_{2,3}y_{3,2}, \quad \Psi_1(\mathbf{y})_j = y_{j,k}y_{j,l}y_{k,j}^2y_{l,j}^2.$$

We note that the coprimality conditions can be expressed as follows: For $(u, v) \in \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$, we have $\gcd(y_u, y_v) = 1$ if and only if the divisors E_u and E_v do not intersect on $\tilde{\mathcal{S}}_1$, which holds if and only if the corresponding rays in the fan of $\tilde{\mathcal{S}}_1$ (Figure 1) are not neighbors.

Since the $y_{k,j}$ are unique up to units in \mathcal{O}_K , the map Ψ_1 induces a w_K^4 -to-1 map from

$$\{\mathbf{y} \in (\mathcal{O}_K^{\neq 0})^6 \mid H(\Psi_1(\mathbf{y})) \leq B, \text{coprimality as in the fan of } \tilde{\mathcal{S}}_1 \text{ in Figure 1}\}$$

to $N_0(B)$.

In the second step, let $y_j = \gcd(y_{k,j}, y_{l,j})$. As before, we obtain a map

$$\Psi_2 : (\mathcal{O}_K^{\neq 0})^9 \rightarrow S(K), \quad \mathbf{y} \mapsto \Psi_2(\mathbf{y}) = (\Psi_2(\mathbf{y})_0 : \dots : \Psi_2(\mathbf{y})_3),$$

where $\mathbf{y} = (y_1, y_2, y_3, y_{1,2}, y_{2,1}, y_{1,3}, y_{3,1}, y_{2,3}, y_{3,2})$ and

$$\Psi_2(\mathbf{y})_0 = y_1y_2y_3y_{1,2}y_{2,1}y_{1,3}y_{3,1}y_{2,3}y_{3,2}, \quad \Psi_2(\mathbf{y})_j = y_j^3y_{j,k}y_{j,l}y_{k,j}^2y_{l,j}^2.$$

This induces a w_K^7 -to-1 map from

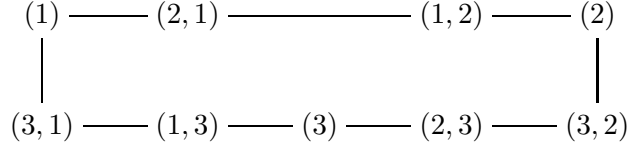
$$\{\mathbf{y} \in (\mathcal{O}_K^{\neq 0})^9 \mid H(\Psi_2(\mathbf{y})) \leq B, \text{coprimality as in the fan of } \tilde{\mathcal{S}} \text{ in Figure 1}\}$$

to $N_0(B)$.

Now we note that

$$H(\Psi_2(\mathbf{y})) = \max\{\|\Psi_2(\mathbf{y})_1\|_{\infty}, \|\Psi_2(\mathbf{y})_2\|_{\infty}, \|\Psi_2(\mathbf{y})_3\|_{\infty}\}$$

because the coprimality conditions imply that $\Psi_2(\mathbf{y})_0, \dots, \Psi_2(\mathbf{y})_3$ are co-prime for any \mathbf{y} satisfying the coprimality conditions, and the archimedean norm of $\Psi_2(\mathbf{y})_0$ cannot be larger than all other three. Indeed, the second observation follows from $\Psi_2(\mathbf{y})_0^3 = \Psi_2(\mathbf{y})_1\Psi_2(\mathbf{y})_2\Psi_2(\mathbf{y})_3$. For the first observation, we note that any prime may divide at most two variables whose corresponding rays in Figure 1 are neighbors, and one checks that for each such pair of variables, there is one monomial in which these variables do not occur.

FIGURE 2. Graph $G = (V, E)$ encoding coprimality conditions.

We reformulate the coprimality conditions as follows, using the graph $G = (V, E)$ with nine vertices $V = \{(1), (2, 1), \dots\}$ and nine edges $E = \{(1), (2, 1)\}, \{(2, 1), (1, 2)\}, \dots\}$ in Figure 2.

Lemma 1. *Let $G = (V, E)$ be the graph in Figure 2. Let E' be the set all pairs $\{u, v\}$ of vertices $u, v \in V$ which are not adjacent in the graph.*

We have

$$N_{U,K,H}(B) = \frac{1}{w_K^7} \sum_{\substack{\mathbf{y} \in (\mathcal{O}_K^{\neq 0})^V \cap M(B) \\ \gcd(y_u, y_v) = 1 \text{ for all } \{u, v\} \in E'}} 1,$$

where $M(B)$ is the set of all $\mathbf{y} \in \mathbb{C}^V$ with

$$\|y_j^3 y_{j,k} y_{j,l} y_{k,j}^2 y_{l,j}^2\|_{\infty} \leq B$$

for all $(j, k, l) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$.

5. MÖBIUS INVERSIONS

Having found a suitable parameterization of K -rational points by points over \mathcal{O}_K in an open subset of \mathbb{A}^9 in Lemma 1, the main problem is essentially to estimate the number of lattice points in the region described by the height conditions. This is done in Lemma 2; its proof is deferred to Section 6. Here, we remove the coprimality conditions by a Möbius inversion and recover the non-archimedean densities.

Applying Möbius inversion over all elements of E' to the expression in Lemma 1 gives

$$N_{U,K,H}(B) = \frac{1}{w_K^7} \sum_{\mathbf{d} \in \mathbb{N}_K^{E'}} \prod_{\alpha \in E'} \mu(d_{\alpha}) \sum_{\substack{\mathbf{y} \in (\mathcal{O}_K^{\neq 0})^V \cap M(B) \\ d_{\{u,v\}} | y_u, y_v \forall \{u,v\} \in E'}} 1.$$

We collect all terms dividing some y_v to obtain

$$N_{U,K,H}(B) = \frac{1}{w_K^7} \sum_{\mathbf{d} \in \mathbb{N}_K^{E'}} \prod_{\alpha \in E'} \mu(d_{\alpha}) \sum_{\substack{\mathbf{y} \in (\mathcal{O}_K^{\neq 0})^V \cap M(B) \\ r_v | y_v \forall v \in V}} 1,$$

where r_v is defined as the lowest common multiple of the d_{α} with $\alpha \in E'$ and $v \in \alpha$. This sum can be estimated as follows; see Section 6 for the proof.

Lemma 2. *For $\mathbf{r} \in \mathbb{N}_K^V$, let*

$$R_1 = \prod_{v \in V} \|r_v\|_{\infty}, \quad R_2 = \prod_{\substack{j,k \in \{1,2,3\} \\ j \neq k}} \|r_{j,k}\|_{\infty}^{2/3} \prod_{j \in \{1,2,3\}} \|r_j\|_{\infty} (\max_j \|r_j\|_{\infty})^{-1/2}.$$

Then

$$\sum_{\substack{\mathbf{y} \in (\mathcal{O}_K^{\neq 0})^V \cap M(B) \\ r_v | y_v \forall v \in V}} 1 = \frac{2^7 \pi^9}{6! d_K^9} \frac{B}{R_1} (\log(B))^6 + O\left(\frac{B}{R_2} (\log(B))^5\right).$$

Combining this with Lemma 1 gives

$$N_{U,K,H}(B) = \frac{2^7 \pi^9}{6! w_K^7 d_K^9} \omega B (\log(B))^6 + O(\rho B (\log B)^5),$$

where

$$\omega = \sum_{\mathbf{d} \in \mathbb{N}_K^{E'}} \prod_{\alpha \in E'} \mu(d_\alpha) \frac{1}{R_1}, \quad \rho = \sum_{\mathbf{d} \in \mathbb{N}_K^{E'}} \prod_{\alpha \in E'} |\mu(d_\alpha)| \frac{1}{R_2}$$

with R_1 and R_2 depending on \mathbf{d} .

To show that ω and ρ are well-defined it will enough to show the convergence of the defining sum of ρ since $|\omega| \leq \rho$. The Euler factor of ρ corresponding to some prime $p \in \mathbb{N}_K$ is $1 + O(\|p\|_\infty^{-7/6})$. Indeed, the factor will have only finitely many non-vanishing summands since $\mu(p^e) = 0$ for all $e \geq 2$. For $\mathbf{d} = (1, \dots, 1)$, we have $R_2 = 1$. For \mathbf{d} with $d_\alpha = p$ for at least one $\alpha = \{u, v\} \in E'$, we have $r_u = r_v = p$ and therefore $R_2 \geq \|r_u\|_\infty^a \|r_v\|_\infty^b \geq \|p\|_\infty^{7/6}$ where $a, b \in \{\frac{1}{2}, \frac{2}{3}, 1\}$, with at most one of them equal to $\frac{1}{2}$.

Let us now calculate the Euler factors A_p of ω for some prime $p \in \mathbb{N}_K$. Let $A \in \mathbb{Z}[x]$ be the polynomial

$$A(x) = \sum_{\tilde{\mathbf{d}} \in \{0,1\}^{E'}} \prod_{\alpha \in E'} \tilde{\mu}(\tilde{d}_\alpha) x^{\sum_{v \in V} \tilde{r}_v}.$$

Here, $\tilde{\mathbf{r}} \in \{0,1\}^V$ is defined depending on $\tilde{\mathbf{d}}$ as follows: For any $v \in V$, the number \tilde{r}_v is the maximum of all \tilde{d}_α with $v \in \alpha$. Furthermore, $\tilde{\mu}$ is defined by

$$\tilde{\mu}(n) = \begin{cases} 1, & n = 0, \\ -1, & n = 1. \end{cases}$$

Then $A_p = A(\|p\|_\infty^{-1})$.

By further Möbius inversions, we have

$$\begin{aligned} A(x) &= \sum_{\tilde{\mathbf{k}} \in \{0,1\}^V} x^{\sum_{v \in V} \tilde{k}_v} \sum_{\substack{\tilde{\mathbf{d}} \in \{0,1\}^{E'} \\ \tilde{\mathbf{r}} = \tilde{\mathbf{k}}}} \prod_{\alpha \in E'} \tilde{\mu}(\tilde{d}_\alpha) \\ &= \sum_{\tilde{\mathbf{n}} \in \{0,1\}^V} \prod_{v \in V} e(\tilde{n}_v) \sum_{\substack{\tilde{\mathbf{d}} \in \{0,1\}^{E'} \\ \tilde{d}_\alpha \leq \tilde{n}_v \text{ if } v \in \alpha}} \prod_{\alpha \in E'} \tilde{\mu}(\tilde{d}_\alpha), \end{aligned}$$

where the function $e : \{0,1\} \rightarrow \mathbb{Q}[x]$ defined by

$$e(n) = \begin{cases} 1 - x, & n = 0, \\ x, & n = 1 \end{cases}$$

is chosen such that

$$\sum_{k \in \{0,1\}} x^k F(k) = \sum_{n \in \{0,1\}} e(n) \sum_{0 \leq s \leq n} F(s)$$

for any function $F : \{0, 1\} \rightarrow \mathbb{Z}$; this is applied above $\#V$ times. We have also used that $\tilde{r}_v \leq \tilde{n}_v$ if $d_\alpha \leq \tilde{n}_v$ for all $\alpha \in E'$ containing $v \in V$.

Note that since $\tilde{\mu}(0) + \tilde{\mu}(1) = 0$ for a fixed $\tilde{\mathbf{n}} \in \{0, 1\}^V$, the sum over $\tilde{\mathbf{d}}$ will vanish if two vertices of $\{v \in V \mid \tilde{n}_v = 1\}$ can be joined by a line in E' . So it will not vanish only if either all \tilde{n}_v are 0, exactly one of the nine \tilde{n}_v is equal to 1 or exactly two \tilde{n}_v, \tilde{n}_w are 1 where $\{v, w\}$ is one of the nine edges E . So we have

$$A(x) = (1-x)^9 + 9(1-x)^8x + 9(1-x)^7x^2 = (1-x)^7 \cdot (1+7x+x^2)$$

and finally

$$\omega = \prod_p (1 - \|p\|_\infty^{-1})^7 (1 + 7\|p\|_\infty^{-1} + \|p\|_\infty^{-2}).$$

Up to the proof of Lemma 2, this completes the proof of our main theorem.

6. ESTIMATIONS OF LATTICE POINTS

In this section, we prove Lemma 2. We proceed as in [HBM99]. First, we rewrite the sum as

$$\sum_{\substack{\mathbf{y} \in (\mathcal{O}_K^{\neq 0})^V \cap M(B) \\ r_v | y_v \forall v \in V}} 1 = \sum_{\substack{\mathbf{z} \in (\mathcal{O}_K^{\neq 0})^V \\ \|z_j\|_\infty \leq C_j}} 1,$$

where we define

$$\zeta_j = z_{j,k} z_{j,l} z_{k,j}^2 z_{l,j}^2, \quad C_j = B^{1/3} \|\zeta_j r_j^3 r_{j,k} r_{j,l} r_{k,j}^2 r_{l,j}^2\|_\infty^{-1/3}$$

for any $\{j, k, l\} = \{1, 2, 3\}$.

From here, unless stated otherwise, we use the convention that whenever j, k appears in a statement, we mean all $j, k \in \{1, 2, 3\}$ with $j \neq k$, and whenever j shows up, we mean all $j \in \{1, 2, 3\}$.

To sum over z_j for $j = 1, 2, 3$, one can use the following estimate on the number of integers in $\mathcal{O}_K^{\neq 0}$ in the circle $\text{Ci}(C) = \{z \in \mathbb{C} \mid \|z\|_\infty \leq C\}$ of radius \sqrt{C} in the complex plane. It seems interesting to note that [HBM99, §4] finds it convenient to use the similar estimate $C + O(\sqrt{C})$ instead of $C + O(1)$ (which is not available in our case) for the number of natural numbers smaller than C .

Lemma 3. *For any positive $C \in \mathbb{R}$, we have*

$$\#(\{x \in \mathcal{O}_K^{\neq 0}\} \cap \text{Ci}(C)) = \frac{2\pi}{d_K} C + O(\sqrt{C}).$$

Proof. The theorem follows in the case $C \geq C_0$ for some $C_0 > 0$ by the theorem on lattice points in homogenously expanding sets since the area of the circle is πC and the area of a fundamental domain is $d_K/2$. The missing point in the origin can be accounted for in the error term in this case. If $C < 1$, then $\text{Ci}(C) \cap \mathcal{O}_K^{\neq 0} = \emptyset$ and we have $(2\pi/d_K)C = O(\sqrt{C})$, so the lemma is also true in this case. Finally, if $1 \leq C \leq C_0$, then $\text{Ci}(C) \subseteq \text{Ci}(C_0)$ and $\sqrt{C} \geq 1$. Therefore, we can choose the implied constant in the error term to be greater than $\#(\text{Ci}(C_0) \cap \mathcal{O}_K^{\neq 0}) + (2\pi/d_K)C_0$, which establishes the lemma in the remaining case. \square

Therefore, with $B_j = B\|r_{j,k}r_{j,l}r_{k,j}^2r_{l,j}^2\|_\infty^{-1}$ for all $\{j, k, l\} = \{1, 2, 3\}$,

$$\begin{aligned} \sum_{\substack{\mathbf{z} \in (\mathcal{O}_K^{\neq 0})^V \\ \|z_j\|_\infty \leq C_j}} 1 &= \sum_{\substack{z_{j,k} \in \mathcal{O}_K^{\neq 0} \\ \|\zeta_j\|_\infty \leq B_j}} \prod_{j=1}^3 \left(\frac{2\pi}{d_K} C_j + O(\sqrt{C_j}) \right) \\ &= \sum_{\substack{z_{j,k} \in \mathcal{O}_K^{\neq 0} \\ \|\zeta_j\|_\infty \leq B_j}} \left\{ \frac{2^3 \pi^3}{d_K^3} C_1 C_2 C_3 + O \left(C_1 C_2 C_3 \max_j (C_j^{-1/2}) \right) \right\} \\ &= \frac{2^3 \pi^3 B}{d_K^3 R_1} \mathcal{M}(B, \mathbf{r}) + O \left(\frac{B^{5/6}}{R_2} \mathcal{R}(B, \mathbf{r}) \right), \end{aligned}$$

where

$$\mathcal{M}(B, \mathbf{r}) = \sum_{\substack{z_{j,k} \in \mathcal{O}_K^{\neq 0} \\ \|\zeta_j\|_\infty \leq B_j}} \prod_{j,k} \|z_{j,k}\|_\infty^{-1}$$

and

$$\mathcal{R}(B, \mathbf{r}) = \sum_{\substack{z_{j,k} \in \mathcal{O}_K^{\neq 0} \\ \|\zeta_j\|_\infty \leq B_j}} \prod_{j,k} \|z_{j,k}\|_\infty^{-1} \max_j \|\zeta_j\|_\infty^{\frac{1}{6}}.$$

Let us begin with the estimation of the error term \mathcal{R} . Because of symmetry it will be no loss to assume that $\|\zeta_1\|_\infty \geq \|\zeta_2\|_\infty, \|\zeta_3\|_\infty$. Then

$$\begin{aligned} \mathcal{R}(B, \mathbf{r}) &= \sum_{\substack{z_{j,k} \in \mathcal{O}_K^{\neq 0} \\ \|\zeta_j\|_\infty \leq B_j}} \|z_{2,1}z_{3,1}\|_\infty^{-2/3} \|z_{2,3}z_{3,2}\|_\infty^{-1} \|z_{1,2}z_{1,3}\|_\infty^{-5/6} \\ &\ll \sum_{\substack{z_{j,k} \in \mathcal{O}_K^{\neq 0} \forall j \neq 1 \\ \|z_{j,k}\|_\infty \leq B \forall j \neq 1}} \|z_{2,1}z_{3,1}\|_\infty^{-2/3} \|z_{2,3}z_{3,2}\|_\infty^{-1} \sum_{\substack{u \in \mathcal{O}_K^{\neq 0} \\ \|u\|_\infty \leq U}} \frac{d(u)}{\|u\|_\infty^{5/6}} \end{aligned}$$

where U is defined as $U = B\|z_{2,1}z_{3,1}\|_\infty^{-2}$ and d is the divisor function in $\mathcal{O}_K^{\neq 0}$.

Here, we need the following auxiliary result, which we will use again later.

Lemma 4. *For all sufficiently large B , we have*

$$\sum_{\substack{x \in \mathcal{O}_K^{\neq 0} \\ \|x\|_\infty \leq B}} \|x\|_\infty^\alpha = \begin{cases} O(B^{\alpha+1}), & -1 < \alpha \leq 0, \\ O(\log B), & \alpha = -1. \end{cases}$$

Proof. For any $n \in \mathbb{N}$, let

$$a_n = \#\{x \in \mathcal{O}_K^{\neq 0} \mid \|x\|_\infty = n\}.$$

By the Abel summation formula, we have

$$\sum_{\substack{x \in \mathcal{O}_K^{\neq 0} \\ \|x\|_\infty \leq B}} \|x\|_\infty^\alpha = \sum_{1 \leq n \leq B} a_n n^\alpha = B^\alpha \sum_{1 \leq n \leq B} a_n - \alpha \int_1^B x^{\alpha-1} \sum_{1 \leq n \leq x} a_n dx.$$

We apply Lemma 3 to the sums over n . The first term is $O(B^{\alpha+1})$. For $-1 < \alpha \leq 0$, the second term is $O(B^{\alpha+1})$ as well; for $\alpha = -1$, it is $O(\log B)$. \square

Using Lemma 4 twice, the inner sum can be estimated elementarily as

$$\begin{aligned} \sum_{\|u\|_\infty \leq U} \frac{d(u)}{\|u\|_\infty^{5/6}} &= \sum_{\|u\|_\infty \leq U} \sum_{v|u} \|u\|_\infty^{-5/6} = \sum_{\|v\|_\infty \leq U} \|v\|_\infty^{-5/6} \sum_{w \leq U \|v\|_\infty^{-1}} \|w\|_\infty^{-5/6} \\ &= O \left(U^{1/6} \sum_{\|v\|_\infty \leq B} \|v\|_\infty^{-1} \right) = O(U^{1/6} \log B). \end{aligned}$$

Inserting this into the original expression for $\mathcal{R}(B, \mathbf{r})$ and applying Lemma 4 again gives

$$\begin{aligned} \mathcal{R}(B, \mathbf{r}) &\ll B^{1/6} \log B \sum_{\substack{z_{j,k} \in \mathcal{O}_K^{\neq 0} \forall j \neq 1 \\ \|z_{j,k}\|_\infty \leq B \forall j \neq 1}} \|z_{2,1} z_{3,1} z_{2,3} z_{3,2}\|_\infty^{-1} \\ &\ll B^{1/6} \log B \left(\sum_{\substack{z \in \mathcal{O}_K^{\neq 0} \\ \|z\|_\infty \leq B}} \|z\|_\infty^{-1} \right)^4 \ll B^{1/6} (\log B)^5. \end{aligned}$$

Now it will be enough to show that

$$\mathcal{M}(B, \mathbf{r}) = \frac{2^4 \pi^6}{6! d_K^6} (\log B)^6 + O(R_3 (\log B)^5)$$

where we define $R_3 = \prod_{j,k} \|r_{j,k}\|_\infty^{1/3}$. Assume this is done for the case of $r_{j,k} = 1$ for all j, k and all $B \geq B_0$ for some $B_0 > 1$. Then on the one hand

$$\mathcal{M}(B, \mathbf{r}) \leq \mathcal{M}(B, (1, \dots, 1)) = \frac{2^4 \pi^6}{6! d_K^6} (\log B)^6 + O((\log B)^5)$$

and on the other hand

$$\mathcal{M}(B, \mathbf{r}) \geq \mathcal{M}(B/R_3^6, (1, \dots, 1)) = \frac{2^4 \pi^6}{6! d_K^6} (\log(B/R_3^6))^6 + O((\log(B/R_3^6))^5)$$

for all \mathbf{r} with $R_3 \leq (B/B_0)^{1/6}$. This gives the required estimate in this case since there is a constant C such that $\log R_3 \leq C R_3^{1/6}$ for any $R_3 \geq 1$. Otherwise we notice that the error term dominates the main term.

It therefore remains to estimate $\mathcal{M}(B) = \mathcal{M}(B, (1, \dots, 1))$. In this case, $B_j = B$.

Lemma 5. *Let*

$$N(B) = \{\mathbf{z} \in \mathbb{C}^6 \mid \|z_{j,k}\|_\infty \geq 1, \|\zeta_j\|_\infty \leq B\},$$

where the ζ_j are defined as before. Define the integral

$$\mathcal{I}(B) = \left(\frac{2}{d_K} \right)^6 \int_{N(B)} \prod_{j,k} \frac{dz_{j,k}}{\|z_{j,k}\|_\infty}.$$

Then $\mathcal{M}(B) = \mathcal{I}(B) + O((\log B)^5)$ for all sufficiently large B .

Proof. We fix a fundamental domain F of the lattice corresponding to \mathcal{O}_K in \mathbb{C} ; its area is $d_K/2$. Our goal is to compare the terms $\prod_{j,k} \|z_{j,k}\|_\infty^{-1}$ of the sum defining $\mathcal{M}(B)$ with integrals over translations of F . For an upper bound for $\mathcal{M}(B)$ compared to $\mathcal{I}(B)$, we must choose a translation $F(z)$ of F whose elements are closer to 0 than $z \in \mathcal{O}_K$. For an upper bound for $\mathcal{I}(B)$ compared to $\mathcal{M}(B)$, we must choose a translation $F'(z)$ of F whose elements are further away from 0 than $z \in \mathcal{O}_K$. Furthermore, we must be careful to stay away from the ball $\|z\|_\infty \leq 1$ and from the real and imaginary axes.

Let R be the smallest rectangle whose sides (of real length l_r resp. imaginary length l_i) are parallel to the real and imaginary axes and that contains F . For any $z \in \mathbb{C}$ with real part $|\Re(z)| \geq 1 + l_r$ (resp. $|\Re(z)| \geq 1$) and imaginary part $|\Im(z)| \geq 1 + l_i$ (resp. $|\Im(z)| \geq 1$), let $R(z)$ (resp. $R'(z)$) be the unique translation of the rectangle R with the following property: The point $z \in \mathbb{C}$ is the corner with the largest (resp. smallest) distance to $0 \in \mathbb{C}$ of $R(z)$ (resp. $R'(z)$). Let $F(z)$ (resp. $F'(z)$) be the unique translation of F contained in $R(z)$ (resp. $R'(z)$). For any $x \in F(z)$ (resp. $x \in F'(z)$), we have $\|z\|_\infty \geq \|x\|_\infty$ (resp. $\|z\|_\infty \leq \|x\|_\infty$).

Let

$$E(B) = \{\mathbf{z} \in N(B) \mid |\Re(z_{j,k})| \geq 1 + l_r, |\Im(z_{j,k})| \geq 1 + l_i\}$$

and $G(B) = N(B) \setminus E(B)$. Let

$$G'(B) = \{\mathbf{z} \in \mathbb{C}^6 \mid 1 \leq \|z_{j,k}\|_\infty \leq B, |\Re(z_{1,2})| \leq 1 + l_r\}.$$

We note that $G(B)$ is contained in the union of $G'(B)$ with eleven other sets of a similar shape (with the analogous condition on $\Re(z_{j,k})$ or $\Im(z_{j,k})$) that we will be able to deal with in the same way as $G'(B)$.

First, we give an upper bound for $\mathcal{M}(B)$ in terms of $\mathcal{I}(B)$. We split $\mathcal{M}(B)$ into a sum over $E(B) \cap (\mathcal{O}_K^{\neq 0})^6$ giving the main term and a sum over $G(B) \cap (\mathcal{O}_K^{\neq 0})^6$ giving the error term. For the main term, we note that the sets $\prod_{j,k} F(z_{j,k})$ for all $\mathbf{z} \in E(B) \cap (\mathcal{O}_K^{\neq 0})^6$ are subsets of $N(B)$ whose pairwise intersections are null sets. As $\|z_{j,k}\|_\infty \geq \|x\|_\infty$ for any $x \in F(z_{j,k})$ with $\mathbf{z} \in E(B) \cap (\mathcal{O}_K^{\neq 0})^6$, we have $\|z_{j,k}\|_\infty^{-1} \leq \frac{2}{d_K} \int_{F(z_{j,k})} \|x\|_\infty^{-1} dx$. Therefore,

$$\sum_{\mathbf{z} \in E(B) \cap (\mathcal{O}_K^{\neq 0})^6} \prod_{j,k} \|z_{j,k}\|_\infty^{-1} \leq \sum_{\mathbf{z} \in E(B) \cap (\mathcal{O}_K^{\neq 0})^6} \prod_{j,k} \frac{2}{d_K} \int_{F(z_{j,k})} \frac{dx_{j,k}}{\|x_{j,k}\|} \leq \mathcal{I}(B).$$

For the error term, we deal with $G'(B)$ instead of $G(B)$, as mentioned before; here,

$$\sum_{\mathbf{z} \in G'(B) \cap (\mathcal{O}_K^{\neq 0})^6} \prod_{j,k} \|z_{j,k}\|_\infty^{-1} \ll (\log B)^5 \sum_{\substack{z_{1,2} \in \mathcal{O}_K^{\neq 0} \\ |\Re(z)| \leq 1 + l_r}} \|z_{1,2}\|_\infty^{-1} \ll (\log B)^5.$$

Indeed, in the first step, we use Lemma 4. In the second step, let N be the maximum number of lattice points in \mathcal{O}_K in a box of real length $1 + l_r$ and imaginary length 1. We note that the sum over $z_{1,2}$ is bounded because all $z_{1,2} \in \mathcal{O}_K^{\neq 0}$ with $|\Re(z_{1,2})| \leq 1 + l_r$ and $|\Im(z_{1,2})| \leq 1$ contribute $\leq 4N$,

and all $z_{1,2} \in \mathcal{O}_K$ with $k \leq |\Im(z_{1,2})| \leq k+1$ contribute $\leq 4Nk^{-2}$ (because $\|z_{1,2}\|_\infty \geq k^2$), which converges when summed over $k \in \mathbb{N}$. In total,

$$\mathcal{M}(B) \leq \mathcal{I}(B) + O((\log B)^5).$$

For the other direction, we note that, for any x with $|\Re(x)| \geq 1 + l_r$ and $|\Im(x)| \geq 1 + l_i$, there is a $z \in \mathcal{O}_K$ such that $x \in F'(z)$, with $|\Re(z)| \geq 1$ and $|\Im(z)| \geq 1$ and $\|z\|_\infty \leq \|x\|_\infty$, by our construction of $F'(z)$. Therefore, for any $\mathbf{x} \in E(B)$, there is a $\mathbf{z} \in N(B) \cap (\mathcal{O}_K^{\neq 0})^6$ such that $\mathbf{x} \in \prod_{j,k} F'(z_{j,k})$. Thus $E(B)$ is covered by $\bigcup_{\mathbf{z} \in N(B) \cap (\mathcal{O}_K^{\neq 0})^6} \prod_{j,k} F'(z_{j,k})$, and

$$\left(\frac{2}{d_K}\right)^6 \int_{E(B)} \prod_{j,k} \frac{dx_{j,k}}{\|x_{j,k}\|_\infty} \leq \sum_{\mathbf{z} \in N(B) \cap (\mathcal{O}_K^{\neq 0})^6} \prod_{j,k} \frac{2}{d_K} \int_{F(z_{j,k})} \frac{dx_{j,k}}{\|x_{j,k}\|_\infty} \leq \mathcal{M}(B).$$

It remains to consider the integral over $G(B)$. Again, we just consider $G'(B)$. Here, we have

$$\begin{aligned} \int_{G'(B)} \prod_{j,k} \frac{dx_{j,k}}{\|x_{j,k}\|_\infty} &\ll (\log B)^5 \int_{1 \leq \|x_{1,2}\|_\infty \leq B, |\Re(x_{1,2})| \leq 1 + l_r} \frac{dx_{1,2}}{\|x_{1,2}\|_\infty} \\ &\ll (\log B)^5 \left(1 + 2(1 + l_r) \int_1^\infty \frac{dx}{x^2}\right) \ll (\log B)^5 \end{aligned}$$

because of $\int_{1 \leq \|x_{j,k}\|_\infty \leq B} \|x_{j,k}\|_\infty^{-1} dx_{j,k} \ll \log B$ and $\|x_{1,2}\|_\infty \geq |\Im(x_{1,2})|^2$ and the boundedness of the integral over all $x_{1,2}$ as above with $|\Im(x_{1,2})| \leq 1$. Therefore,

$$\mathcal{I}(B) \leq \mathcal{M}(B) + O((\log B)^5),$$

completing the proof. \square

It remains to evaluate $\mathcal{I}(B)$. Using the rotation symmetries of its integrands, one can write

$$\mathcal{I}(B) = \left(\frac{2}{d_K}\right)^6 \pi^6 \int \frac{dz_{1,2}}{z_{1,2}} \cdots \frac{dz_{3,2}}{z_{3,2}},$$

where the integral runs now over all real $z_{j,k} \geq 1$ satisfying the three inequalities $z_{j,k} z_{j,l} z_{k,j}^2 z_{l,j}^2 \leq B$ for $\{j, k, l\} = \{1, 2, 3\}$. Substituting $z_{j,k} = B^{t_{j,k}}$ shows that

$$\mathcal{I}(B) = V \frac{2^6 \pi^6}{d_K^6} (\log B)^6,$$

where V denotes the integral

$$V = \int dt_{1,2} \cdots dt_{3,2}$$

over the six-dimensional convex polytope defined by the six inequalities $t_{j,k} \geq 0$ and the three inequalities

$$t_{j,k} + t_{j,l} + 2t_{k,j} + 2t_{l,j} \leq 1$$

for all $\{j, k, l\} = \{1, 2, 3\}$. The volume of this polytope is $V = (4 \cdot 6!)^{-1}$ [HBM99]. This completes the proof of Lemma 2.

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